J. Fluid Mech. (1994), vol. 275, pp. 147–155

A formula for 'wave damping' in the drift of a floating body

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(Received 21 February 1990 and in revised form 14 February 1994)

A simple formula for 'wave damping' is derived, exact within the context of the proposed theory, namely: potential flow correct to second order in the wave amplitude and to leading order in U/c, where U is the drift velocity and c the wave celerity. The analysis is restricted to a two-dimensional problem although the extension to three dimensions seems possible.

1. Introduction

Ocean structures exposed to waves are excited by second-order forces and drift slowly in time. Viscous dissipation is very small, since it is proportional to drift velocity squared, and a second mechanism for damping has been analysed in the last ten years: the influence of drift velocity on the exciting forces that cause it.

This effect is called 'wave damping', after Wichers (1982), and it has been shown that it is the dominant damping mechanism mainly in a high sea state. Wichers (1982) proposed a heuristic formula and Faltinsen (1988) formulated the exact problem under the following assumptions: potential flow correct to second order in the wave amplitude and to leading order in U/c, where U is the drift velocity and c the wave celerity. More recently, Sclavounos (1989) derived an asymptotic simplification of Faltinsen's approach but, in any circumstance, the numerical computation is relatively complex, as discussed briefly in §2 of the present work. Few numerical results of this theory are known and two of them will be discussed here.

In this paper it is shown that, in two dimensions, the exciting drift force, influenced by the drift velocity, is given in deep water by

$$D(\omega; U) = \tfrac{1}{2}\rho g A^2 |R(\omega_e)|^2 [1-4U/c]. \tag{1}$$

In (1) ρ is the water density, g is the acceleration due to gravity, A is the wave amplitude, $c = \omega/K$ is the wave celerity, U is the drift velocity, $\omega_e = \omega - KU$ is the 'frequency of encounter' and $R(.)$ is the reflection coefficient in the standard radiation–diffraction problem (namely, the one where the drift velocity is zero). Formula (1) is exact within the context of the theory proposed by Faltinsen.

As discussed at the end of §2, formula (1) was suggested by an argument based on the theory of wave groups in a moving medium, as expounded in Bretherton & Garret (1969) and Whitham (1974); its mathematical demonstration is given in §3 and §4 presents a comparison between (1) and numerical results together with some concluding remarks.

2. Statement of the problem and some basic results

Consider a two-dimensional body, placed in the plane (y, z) with z being the vertical axis positive upwards, moving in the positive y-direction with a forward speed U and

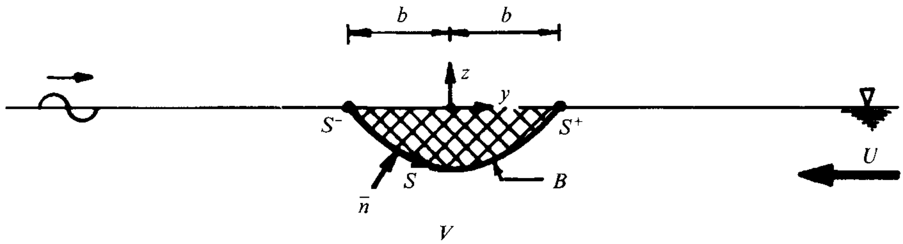

FIGURE 1. Geometry and flow conditions in the moving system.

exposed to a wave with frequency ω, propagating from left to right. In the reference system moving with the forward speed U the observed frequency, named the 'frequency of encounter', is given by the expression

$$\omega_e = \omega(1-U/c); \quad c = g/\omega, \tag{2a}$$

where c is the wave celerity. In this moving system one also observes a current from right to left, as shown in figure 1, and for future reference the following wavenumbers are introduced here:

$$K_e = \omega_e^2/g, \quad K^{\pm} = K_e(1 \pm 2U/c). \tag{2c}$$

The forward velocity U corresponds, in an actual problem, to the drift velocity of the body under wave action and it is small compared with the wave celerity c. Under this condition one can ignore terms of order $(U/c)^2$ and so K^+ is the wavenumber of the incident wave, namely

$$K^+ = \omega^2/g. \tag{2c}$$

If g is the acceleration due to the gravity and A the wave amplitude, the potential of the incident wave, in the moving system, is given by the well-known expression

$$\phi_I(y,z,t) = -\mathrm{i}\frac{gA}{\omega}\,\mathrm{e}^{\mathrm{i}K^+y}\,\mathrm{e}^{-\mathrm{i}\omega_e t}. \tag{3}$$

In what follows the term $\mathrm{e}^{-\mathrm{i}\omega_e t}$ will be factored out and omitted, apart from a few occasions where it may clarify an expression.

Suppose first the interaction between the body and the flow $-U\boldsymbol{j}$ in the absence of waves and let $U\phi_S(y,z)$ be the related potential; the linear free-surface condition is given by $\partial\phi_S/\partial z = U^2\,\partial^2\phi_S/\partial y^2$ and, since terms of order U^2 are to be ignored, it reduces to the 'impermeable' condition $\partial\phi_S/\partial z = 0$. The potential $U\phi_S(y,z)$ describes, then, the distortion caused on the incoming flow by the 'double body' immersed in an unbounded fluid; as a consequence of it the points $S^{\pm}$ shown in figure 1 are 'stagnation points' and since $U\nabla\phi_S \to -U\boldsymbol{j}$ when $|y| \to \infty$ one has

$$\nabla\phi_S(\pm b, 0) = \mathbf{0}, \quad \frac{\partial\phi_S}{\partial y}(\pm\infty, z) = -1. \tag{4}$$

Let $\phi_U(y,z)\,\mathrm{e}^{-\mathrm{i}\omega_e t}$ be the perturbation caused on $U\phi_S(y,z)$ by the action of the wave; the subscript $_U$ indicates a variable related to the oscillatory problem when $U \neq 0$. The total potential can be written in the form

$$\Phi(y,z,t) = U\phi_S(y,z) + \phi_U(y,z)\,\mathrm{e}^{-\mathrm{i}\omega_e t}, \tag{5a}$$

and from Bernoulli's equation the oscillating pressure, linear in ϕ_U, is given by

$$p_U(y,z) = \rho(\mathrm{i}\omega_e\phi_U - U\nabla\phi_S\cdot\nabla\phi_U). \tag{5b}$$

Denoting by $\boldsymbol{n}$ the normal to B pointing out of the fluid, the kinematic condition on the body surface B can be written in terms of the coefficients

$$\begin{aligned} n_2 &= \boldsymbol{n}\cdot\boldsymbol{j}; & m_2 &= -U(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\partial\phi_S/\partial y;\\ n_3 &= \boldsymbol{n}\cdot\boldsymbol{k}; & m_3 &= -U(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\partial\phi_S/\partial z;\\ n_4 &= yn_3-zn_2; & m_4 &= U[-n_2\,\partial\phi_S/\partial z+n_3\,\partial\phi_S/\partial y]+ym_3-zm_2 \end{aligned} \tag{6}$$

and in terms of the generalized displacements $\{q_{j,U}; j=2,3,4\}$ in sway, heave and roll. If $T(\omega_e;U)$ and $R(\omega_e;U)$ are, respectively, the *total* transmission and reflection coefficients then, to leading order in (U/c), the potential $\phi_U(y,z)$ satisfies the set of equations (see Faltinsen & Zhao 1989)

$$\begin{aligned} &\nabla^2\phi_U = 0 \quad \text{in} \quad V;\\ &\frac{\partial\phi_U}{\partial z} = K_e\phi_U + \mathrm{i}\frac{U}{c}\left[2\frac{\partial\phi_S}{\partial y}\frac{\partial\phi_U}{\partial y}+\frac{\partial^2\phi_S}{\partial y^2}\phi_U\right] \quad \text{in} \quad z=0;\\ &\boldsymbol{\nabla}\phi_U\cdot\boldsymbol{n} = \sum_{j=2}^{4} q_{j,U}(-\mathrm{i}\omega_e n_j+m_j) \quad \text{in} \quad B;\\ &\partial\phi_U/\partial z \to 0 \quad \text{for} \quad z\to-\infty;\\ &\phi_U \to -\mathrm{i}\frac{gA}{\omega}\begin{cases} T(\omega_e;U)\,\mathrm{e}^{\mathrm{i}K^+y}\,\mathrm{e}^{K^+z};\\ \mathrm{e}^{\mathrm{i}K^+y}\,\mathrm{e}^{K^+z}+R(\omega_e;U)\,\mathrm{e}^{-\mathrm{i}K^-y}\,\mathrm{e}^{K^-z}. \end{cases} \quad (y\to\pm\infty). \end{aligned} \tag{7}$$

The oscillatory fluid force coefficients can be defined by the expressions

$$\mathscr{F}_{j,U} = \frac{1}{\rho}\int_B p_U(y,z)\,n_j\,\mathrm{d}B = \int_B (\mathrm{i}\omega_e n_j+m_j)\,\phi_U\,\mathrm{d}B, \tag{8}$$

with $p_U(y,z)$ given by (5*b*); notice that $\mathscr{F}_{j,U}$ includes both the exciting and reacting hydrodynamics forces, the latter normally expressed in terms of the added mass and radiation damping matrices; the expression on the right-hand side of (8) is a known result introduced by Ogilvie & Tuck (1969).

The body's motion can be determined from its dynamic equations and fluid forces; if M_{jl} and C_{jl} are, respectively, the inertia and restoring matrices of the body, these equations can be written in the form

$$D_{jl} = -\omega^2 M_{jl}+C_{jl}, \quad \sum_l D_{jl}\,q_{l,U} = \rho\mathscr{F}_{j,U}, \tag{9a}$$

with the matrix D_{jl} being symmetric; in this case the energy relation

$$\mathrm{Im}\left(\sum_j \mathscr{F}^*_{j,U}\,q_{j,U}\right) = 0 \tag{9b}$$

can be easily proven, with * denoting the complex conjugate.

From (7) some basic conservation equations, of energy and linear momentum, can be deduced; to make more direct the exposition they will be stated in the following but are proven in the Appendix.

The energy equation, similar to $|T(\omega)|^2+|R(\omega)|^2 = 1$ in the standard problem, here takes the form

$$|T(\omega_e;U)|^2+\frac{|R(\omega_e;U)|^2}{1-4(U/c)} = 1 \tag{10a}$$

and reduces to the standard expression when $U=0$. The drift force, analogous to $D(\omega) = \frac{1}{2}\rho gA^2|R(\omega)|^2$, is given now by

$$D(\omega;U) = \tfrac{1}{4}\rho gA^2[(1-|T(\omega_e;U)|^2)(1-4U/c)+|R(\omega_e;U)|^2]; \tag{10b}$$

notice that, for convenience, the frequency ω, as measured in the Earth-fixed system, has been used to describe the dependence of $D(.\,;U)$ on the frequency.

To this point no new result has been introduced and related expressions can be found, for instance, in Grue & Palm (1985). The set of equations (7), (8) and (9) must be solved to determine $D(\omega;U)$ by direct pressure integration on the body surface or else by using (10*b*). This is the route taken in Faltinsen & Zhao (1989), for example, and the numerical work is, in general, complex; one must first compute $\phi_S(y,z)$ and later solve (7) with the relatively complicated free-surface boundary condition shown. However, as it will be seen next, a reasonably plausible assumption can be introduced and it discloses the argument that leads to formula (1).

The basic idea is the following: once terms of order U^2 are ignored the free surface becomes 'impermeable' for the potential $\phi_S(y,z)$ and the interaction between the body and the incoming flow should not generate *new* waves; as a corollary, far-field observers detect only the interaction between the flow $-U\boldsymbol{j}$ with the far-field waves and, using known results of propagation of wave groups in a moving medium, they are able to predict how this interaction takes place.

To make this argument more explicit one may consider, initially, the problem (7) with $U=0$, where the ratio of the transmitted to incident wave amplitudes is given by $|T(\omega_e)|$. If the current is now 'turned on' and increases gradually from zero to its final value U, far-field observers can follow the interaction between these waves and the current. From the theory it is known that such interaction depends only on the ratio U/c and on the relative directions of the wave and current; but then the rates of increase of the incident and transmitted wave amplitudes are the same and so the transmission coefficient should remain constant as the flow intensity increases from zero to U. Using this result in the energy relation (10*a*) one obtains

$$\begin{aligned} |T(\omega_e;U)|^2 &= |T(\omega_e)|^2, \\ |R(\omega_e;U)|^2 &= |R(\omega_e)|^2(1-4U/c). \end{aligned} \tag{11}$$

Now placing expression (11) into (10*b*), formula (1) can be derived directly; this simple result will be formally demonstrated in the following.

3. Derivation of (11)

The demonstration of formula (1) is, as seen above, a consequence of (11); in this section this latter expression will be derived. The technique to be used is based on Green's identity and it is standard in hydrodynamics.

In order to proceed one should consider first, as suggested at the end of the last section, problem (7) when $U=0$; reserving the subscript $_0$ to indicate the related variables, the potential $\phi_0(y,z)$ satisfies the standard set of equations

$$\begin{aligned} &\nabla^2\phi_0 = 0 \quad \text{in} \quad V; \\ &\partial\phi_0/\partial z = K_e\phi_0 \quad \text{in} \quad z=0; \\ &\boldsymbol{\nabla}\phi_0\cdot\boldsymbol{n} = \sum_{j=2}^{4} q_{j,0}(-\mathrm{i}\omega_e n_j) \quad \text{in} \quad B; \\ &\partial\phi_0/\partial z \to 0 \quad \text{for} \quad z\to-\infty; \\ &\phi_0 \to -\mathrm{i}\frac{gA}{\omega}\begin{cases} T(\omega_e)\,\mathrm{e}^{\mathrm{i}K_e y}\,\mathrm{e}^{K_e z}; \\ \mathrm{e}^{\mathrm{i}K_e y}\,\mathrm{e}^{K_e z} + R(\omega_e)\,\mathrm{e}^{-\mathrm{i}K_e y}\,\mathrm{e}^{K_e z} \end{cases} \quad (y\to\pm\infty). \end{aligned} \tag{12a}$$

In this case the fluid force coefficients can be written as

$$\mathscr{F}_{j,0} = \int_B (\mathrm{i}\omega_e n_j)\,\phi_0\,\mathrm{d}B \tag{12b}$$

while the dynamic equations are given by

$$\sum_j D_{jl}\,q_{l,0} = \rho\mathscr{F}_{j,0}. \tag{12c}$$

From the symmetry of D_{jl} the identity

$$\sum_j \mathscr{F}_{j,U}\,q_{j,0} = \sum_j \mathscr{F}_{j,0}\,q_{j,U} \tag{12d}$$

can be demonstrated; notice that (12*d*) is just one of several possible 'reciprocity relations' in a self-adjoint system.

Consider now the equality $\nabla^2\phi_U\,\phi_0 = \nabla^2\phi_0\,\phi_U$ at every point of the fluid region V, bounded by the free surface F and the body B, and let $\mathscr{V}$ be defined by the expression

$$\mathscr{V} = \sum_j q_{j,U}\int_B (-\mathrm{i}\omega_e n_j + m_j)\,\phi_0\,\mathrm{d}B + \sum_j q_{j,0}\int_B (\mathrm{i}\omega_e n_j)\,\phi_U\,\mathrm{d}B. \tag{13a}$$

If the above equality is integrated in V and Green's theorem is used one obtains

$$\mathscr{V} + \mathrm{i}\frac{U}{c}\int_F \left[2\frac{\partial\phi_S}{\partial y}\frac{\partial\phi_U}{\partial y}\phi_0 + \frac{\partial^2\phi_S}{\partial y^2}\phi_U\phi_0\right]_{z=0}\mathrm{d}y = \int_{-\infty}^{0}\left[\frac{\partial\phi_0}{\partial y}\phi_U - \phi_0\frac{\partial\phi_U}{\partial y}\right]_{y=-\infty}^{y=+\infty}\mathrm{d}z.$$

Using in this last integral the far-field expressions for ϕ_U and ϕ_0, as shown in (7) and (12*a*), and recalling that $K^{\pm} = K_e(1\pm 2U/c)$, see (2*b*), the following identity is derived:

$$\begin{aligned}\mathscr{V} &+ \mathrm{i}\frac{U}{c}\int_F \left[2\frac{\partial\phi_S}{\partial y}\frac{\partial\phi_U}{\partial y}\phi_0 + \frac{\partial^2\phi_S}{\partial y^2}\phi_U\phi_0\right]_{z=0}\mathrm{d}y \\ &= -\mathrm{i}\frac{U}{c}2K_e\int_{-\infty}^{0}[\phi_0\phi_U]_{y=-\infty}^{y=+\infty}\mathrm{d}z + 2\mathrm{i}K_e\,\mathrm{e}^{2\mathrm{i}(U/c)K_e y}\left[\frac{R(\omega_e)}{K_e+K^+} - \frac{R(\omega_e;U)}{K_e+K^-}\right].\end{aligned}$$

Introducing the notation

$$\mathscr{P} = \mathrm{i}\frac{U}{c}\left[\int_F \left[2\frac{\partial\phi_S}{\partial y}\frac{\partial\phi_U}{\partial y}\phi_0 + \frac{\partial^2\phi_S}{\partial y^2}\phi_U\phi_0\right]_{z=0}\mathrm{d}y + 2K_e\int_{-\infty}^{0}[\phi_0\phi_U]_{y=-\infty}^{y=+\infty}\mathrm{d}z\right] \tag{13b}$$

one obtains, with the help of (2*b*),

$$\frac{R(\omega_e)}{2+2(U/c)} - \frac{R(\omega_e;U)}{2-2(U/c)} = -\tfrac{1}{2}\mathrm{i}\,[\mathscr{V}+\mathscr{P}]\,\mathrm{e}^{-2\mathrm{i}(U/c)K_e y}. \tag{13c}$$

By definition $\phi_U(y,z)\to\phi_0(y,z)$ when $U\to 0$ and since terms of order $(U/c)^2$ have been consistently ignored the difference $\phi_U-\phi_0$ is of order U/c; one may state this point explicitly by writing

$$\begin{aligned}\phi_U(y,z) &= \phi_0(y,z) + (U/c)\,\delta\phi_0(y,z),\\ q_{j,U} &= q_{j,0} + (U/c)\,\delta q_{j,0},\end{aligned} \tag{14}$$

where the variations ('derivatives') $\delta\phi_0$ and $\delta q_{j,0}$ are of order 1. Placing these expressions in (13*b*), disregarding terms of order $(U/c)^2$ and observing that $\phi_0^2(\pm\infty,z) = \phi_0^2(\pm\infty,0)\,\mathrm{e}^{2K_e z}$, one obtains

$$\mathscr{P} = \mathrm{i}\frac{U}{c}\left[\int_F \frac{\partial}{\partial y}\left[\frac{\partial\phi_S}{\partial y}\phi_0^2\right]\mathrm{d}y + [\phi_0^2(+\infty,0) - \phi_0^2(-\infty,0)]\right],$$

and so $\mathscr{P} = 0$ since $\phi_S(y,z)$ satisfies (4).

A similar result can be derived for $\mathscr{V}$. In fact, observing that $m_j \approx O(U)$, then to leading order in U/c one has

$$\mathscr{F}_{j,U} = \int_B (\mathrm{i}\omega_e n_j + m_j)\,\phi_U\,\mathrm{d}B = \int_B (\mathrm{i}\omega_e n_j)\,\phi_U\,\mathrm{d}B + \int_B m_j\,\phi_0\,\mathrm{d}B,$$

$$\Sigma q_{j,U} \int_B m_j\,\phi_0\,\mathrm{d}B = \Sigma q_{j,0} \int_B m_j\,\phi_0\,\mathrm{d}B;$$

placing these expressions in (13a) one obtains, with the help of (12b) and (12d):

$$\mathscr{V} = \Sigma \mathscr{F}_{j,U}\, q_{j,0} - \Sigma \mathscr{F}_{j,0}\, q_{j,0} = 0.$$

From (13c) it follows then that $|R(\omega_e; U)| = (1-2U/c)\,|R(\omega_e)|$ and so (11) is obtained with the help of the energy equation (10a); this demonstrates formula (1).

4. Numerical results and conclusions

Few numerical results on wave damping are known and two of them will be discussed here. Faltinsen (1988) and Faltinsen & Zhao (1989), for example, analysed only one problem: a semicircle in deep water, as shown in figure 2. In this case the potential $\phi_S(y,z)$ has a simple analytical expression (an uniform flow plus a y-dipole) and so only $\phi_U(y,z,t)$ was computed. To use formula (1) the values of $\{D(\omega_n); \omega_n(d/g)^{1/2} = 0.05n\}$, for $n = 10, \ldots, 26$, were taken from Faltinsen's curve $U = 0$ and the values of $|R(\omega_e)|^2$ were determined by linear interpolation. The agreement between Faltinsen's numerical work and formula (1) is evident.

In order to discuss the second result it is worthwhile to make a digression with respect to formula (1) that has interest in itself. In fact, observing that the drift force in the standard problem is given by $D(\omega) = \frac{1}{2}\rho g A^2 |R(\omega)|^2$, one can write (1) in the form

$$D(\omega; U) = D(\omega_e)\,[1-4(U/c)].$$

Now, disregarding terms of order $(U/c)^2$, one has $D(\omega_e) = D(\omega) - (\partial D/\partial\omega)\,\omega U/c$, since $\omega_e = \omega(1-U/c)$; it turns out then that the force $D(\omega; U)$ can be written as $(c = g/\omega)$

$$\begin{aligned} D(\omega; U) &= D(\omega) - B(\omega)\,U, \\ B(\omega) &= \frac{\omega^2}{g}\frac{\partial D}{\partial \omega}(\omega) + 4\frac{\omega}{g} D(\omega). \end{aligned} \qquad (15)$$

Notice that the forcing function $D(\omega; U)$ has been decomposed into the drift force $D(\omega)$ for the standard problem plus a parcel $-B(\omega)\,U$, proportional to the drift velocity U; for this reason the term $B(\omega)$ is called the 'wave damping coefficient'.

Wichers (1982) proposed an expression for $B(\omega)$ similar to (15) but without the term $4(\omega/g)\,D(\omega)$ and his expression has been tested against numerical results in some simple configurations.

In fact, Clark, Malenica & Molin (1992) compared numerical results for a vertical circular cylinder with Wichers' expression, when the waves and forward speed are collinear. They observed a poor agreement and to improve the result they added, apparently by chance, the second term in (15). They then found an exact fit between

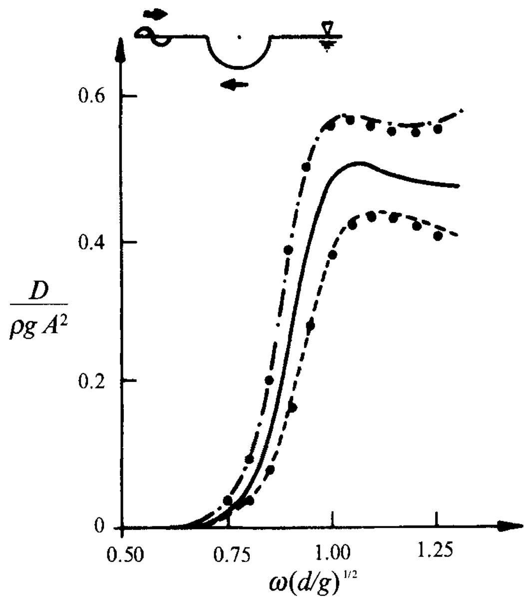

FIGURE 2. Numerical results obtained by Faltinsen (1988) compared with (1) for a semicircle restrained in a roll. Faltinsen: $-\cdot-\cdot-$, $U/(gd)^{1/2}=-0.032$; ——, $U=0$; ----, $U(gd)^{1/2}=0.032$. Formula (1): ●.

(15) and the numerical results. These authors reported, however, that the agreement was a little poorer when the cylinder is free to oscillate, a difference not supported by the present theoretical result.

Clearly (15) has been proven in the context of a two-dimensional theory and one should not expect that it should be exact for a three-dimensional problem. However, the agreement in this particular case, and also in a very similar looking expression proposed by Clark *et al.* when the waves and speed are not collinear, seem to indicate that indeed an expression similar to (15) can be derived for a three-dimensional problem.

The author acknowledges several suggestions made by an associate editor of JFM during the review of the paper; he is also indebted to some imaginary discussions with Professor C. C. Mei, from MIT.

Appendix

In §2 of this work some of the more technical results, for example (10*a*) and (10*b*), were stated without proof and used; they will be demonstrated here.

A.1. *Derivation of (10a)*

Consider the equality $\phi_U^*\nabla^2\phi_U=\phi_U\nabla^2\phi_U^*$ in V, with $\phi_U(y,z)$ being the solution of (7); by partial integration and Green's theorem one has

$$\sum_j(\mathscr{F}^*_{j,U}q_{j,U}-\mathscr{F}_{j,U}q^*_{j,U})+\mathrm{i}(U/c)\int_F \partial/\partial y\,[(\partial\phi_s/\partial y)\,|\phi_U|^2]\,\mathrm{d}B$$
$$=\int_{-\infty}^{0}[(\partial\phi_U^*/\partial y)\,\phi_U-(\partial\phi_U/\partial y)\,\phi_U^*]_{y=-\infty}^{y=+\infty}\,\mathrm{d}y.$$

Using (4), (9*b*) and the far-field behaviour of (7) one obtains, after some algebra,

$$(1-2(U/c))\,|T(\omega_e;U)|^2+(1+2(U/c))\,|R(\omega_e;U)|^2 = 1-2(U/c). \qquad \text{(A 1)}$$

Expression (10*a*) follows at once if terms of order $(U/c)^2$ are disregarded.

A.2. *Derivation of (10b)*

The potential $\Phi(y,z,t)$, defined in (5*a*), can be written in the far field in the form

$$\left.\begin{aligned}\Phi(y,z,t) &= -Uy+\Phi_U(y,z,t),\\ \Phi_U(y,z,t) &= \tfrac{1}{2}[\phi_U(y,z)\,\mathrm{e}^{-\mathrm{i}\omega_e t}+\phi_U^*(y,z)\,\mathrm{e}^{\mathrm{i}\omega_e t}],\end{aligned}\right\} \qquad \text{(A 2a)}$$

with $\phi_U(y,z)$ being the solution of (7). The pressure is then given by

$$p(y,z,t) = -\rho(\partial\Phi_U/\partial t)+\rho U(\partial\Phi_U/\partial y)-\tfrac{1}{2}\rho|\nabla\Phi_U|^2-\rho gz \qquad \text{(A 2b)}$$

and the free-surface displacement $\eta(y,t)$ can be expressed, correct to leading order in the wave amplitude, as

$$\eta(y,t) = (1/g)(-\partial\Phi_U/\partial t+U\,\partial\Phi_U/\partial Y)_{z=0}. \qquad \text{(A 2c)}$$

Let $\langle\cdot\rangle$ be the time-average operator; from momentum considerations (see, for instance, Maruo 1960 or Grue & Palm 1985) the steady drift force $D(\omega;U)$ is given by

$$D(\omega;U) = \langle\mathscr{Q}(-\infty,t)\rangle-\langle\mathscr{Q}(+\infty,t)\rangle, \qquad \text{(A 3a)}$$

with $\mathscr{Q}(y,t)$ defined as

$$\mathscr{Q}(y,t) = \int_{-\infty}^{\eta}[\rho(\partial\Phi_U/\partial y+v_{2,U}-U)^2+p]\,\mathrm{d}z. \qquad \text{(A 3b)}$$

In (A 3*b*) $v_{2,U}(y,z,t)$ is the second-order particle velocity, a term that should *a priori* be considered in the present analysis; placing (A 2*b*) and (A 2*c*) into (A 3*b*) and disregarding terms of cubic order in the wave amplitude, one obtains

$$\begin{aligned}\langle\mathscr{Q}(y,t)\rangle &= -2\rho U\int_{-\infty}^{0}\langle v_{2,U}\rangle\,\mathrm{d}z\\ &\quad+\frac{\rho}{g}\left\langle\frac{1}{2}\left(\frac{\partial\Phi_U}{\partial t}\right)^2+U\left(\frac{\partial\Phi_U}{\partial y}\right)\left(\frac{\partial\Phi_U\partial t}{}\right)\right\rangle_{z=0}+\frac{1}{2\rho}\left\langle\int_{-\infty}^{0}\left[\left(\frac{\partial\Phi_U}{\partial y}\right)^2-\left(\frac{\partial\Phi_U}{\partial z}\right)^2\right]\mathrm{d}z\right\rangle.\end{aligned}$$

Let $\mathscr{M}_0(y)$ be the steady flux of mass (mass transport) through the section y in the standard problem, defined in (12*a*); then

$$\mathscr{M}_0(y) = \rho\int_{-\infty}^{0}\langle v_{2,0}(y,z,t)\rangle\,\mathrm{d}z. \qquad \text{(A 4)}$$

With this notation and ignoring terms of order $(U/c)^2$ one can write, with the help of (A 2*a*)

$$\begin{aligned}\langle\mathscr{Q}(y,t)\rangle &= -2U\mathscr{M}_0(y)\\ &\quad+\tfrac{1}{4}\rho\left[K_e|\phi_U|^2+\frac{U}{c}\left(\mathrm{i}\phi_U^*\frac{\partial\phi_U}{\partial y}+(*)\right)\right]_{z=0}+\tfrac{1}{4}\rho\int_{-\infty}^{0}\left(\left|\frac{\partial\phi_U}{\partial y}\right|^2-\left|\frac{\partial\phi_U}{\partial z}\right|^2\right)\mathrm{d}z. \qquad \text{(A 5)}\end{aligned}$$

From mass conservation one should expect $\mathrm{d}\mathscr{M}_0/\mathrm{d}y = 0$; in fact, for the transmitted wave one obtains the classical results $\mathscr{M}_0(+\infty) = (c/4K)(KA)^2|T(\omega_e)|^2$, see, for instance, Dean & Eagleason (1965), while in the opposite limit it is not difficult to show that $\mathscr{M}_0(-\infty) = (c/4K)(KA)^2[1-|R(\omega_e)|^2] = \mathscr{M}_0(+\infty)$. So the first parcel in (A 5) has no contribution for the steady drift and will be ignored in the following.

Using the far-field condition of (7) one obtains, when $y \rightarrow \infty$,

$$\langle \mathscr{Q}(+\infty, t)\rangle = \tfrac{1}{4}\rho(gA/\omega)^2 K^- |T(\omega_e; U)|^2$$
$$= \tfrac{1}{4}\rho g A^2 |T(\omega_e; U)|^2 (1-4U/c), \tag{A 6a}$$

since $K^+ = \omega^2/g$, see (2c); a similar analysis when $y \rightarrow -\infty$ gives

$$\langle \mathscr{Q}(-\infty, t)\rangle = \tfrac{1}{4}\rho g A^2[(1-4U/c) + |R(\omega_e; U)|^2]. \tag{A 6b}$$

Placing (A 6) in (A 3a) one obtains (10b).

REFERENCES

Bretherton, F. P. & Garret, C. J. R. 1969 Wave trains in inhomogeneous moving media. *Proc. R. Soc. Lond.* A **302**, 529–554.

Clark, P. J., Malenica, S. & Molin, B. 1992 An heuristic approach to wave drift damping. *Appl. Ocean Res.* **15**, 53–55.

Dean, R. G. & Eagleason, P. S. 1965 Finite amplitude waves. In *Estuary and Coastline Hydrodynamics* (ed. A. T. Ippen), pp. 93–132.

Faltinsen, O. M. 1988 Second order nonlinear interaction between waves and low frequency body motion. *IUTAM Symp. on Nonlinear Water Waves, Tokyo* (ed. K. Horikawa & H. Maruo). Springer.

Faltinsen, O. M. & Zhao, R. 1989 Slow-drift motion of a moored two-dimensional body in irregular waves. *J. Ship Res.* **33**, 93–106.

Grue, J. & Palm, E. 1985 Wave radiation and wave diffraction from a submerged body in uniform current. *J. Fluid Mech.* **151**, 257–278.

Maruo, H. 1960 The drift of a body floating in waves. *J. Ship Res.* December 1960, 1–10.

Ogilvie, T. F. & Tuck, E. O. 1969 A rational strip theory of ship motions: Part I. *Rep.* N. 013. The Department of Naval Architecture and Marine Engineering, College of Engineering, The University of Michigan, Ann Arbor.

Sclavounos, P. D. 1989 The slow drift wave damping of floating bodies. *4th Intl. Workshop on Water Waves and Floating Bodies, Norway* (ed. J. Grue). Department of Mathematics, University of Oslo.

Whitham, G. B. 1974 *Linear and Nonlinear Waves.* Wiley-Interscience.

Wichers, J. E. W. 1982 On the low-frequency surge motions of vessels moored in high seas. *14th Annual OTC, Houston, Texas.*